Two Novel Tests for the Rationality of Square Roots

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Abstract

This paper introduces two original methods for determining the rationality or irrationality of square roots of natural numbers. The first method is an algebraic transformation that recursively applies the quadratic formula to test for rational or complex roots. The second method leverages logarithmic bounding to estimate surrounding perfect squares. Both methods provide unique insights into the behavior of square roots and serve as alternatives to traditional contradiction-based number theory techniques.

Introduction

Determining whether $\sqrt{n}$ is rational or irrational is a classical question in number theory. Traditional proofs often involve contradiction or prime factorization. This paper presents two intuitive and original methods developed by the author. The first method transforms the square root into a quadratic equation and examines its discriminant recursively. The second method uses a logarithmic formula to determine the two perfect squares that surround a number and derives conclusions from their structure.

Method 1

Let $e \in \mathbb{N}$. Suppose the square root of e can be written as:

$$\sqrt{e} = x + e$$

This expression is deliberately designed not to be algebraically rearranged universally to solve for x in terms of e, as such attempts lead back to the original form. Instead, the idea is to substitute a specific value for e and solve for x using the quadratic formula.

Procedure

1. Choose a specific natural number e.
2. Write the equation: $\sqrt{e} = x + e$
3. Square both sides: $e = x^2 - 2ex + e^2$
4. Rearranged into quadratic form in x:

$$x^2 + 2ex + (e^2 - e) = 0$$

5. Apply the quadratic formula:

$$x = \frac{-2e \pm \sqrt{4e^2 - 4(e^2 - e)}}{2}$$

Simplify the discriminant:

$$D = 4e^2 - 4(e^2 - e) = 4e$$

$$x = -e \pm \sqrt{e}$$

If this result yields a new square root that cannot be solved (like $\sqrt{e}$), the process could also be finished here and declared irrational, because of the recursive nature of the square roots; similarly, it can continue recursively to eventually meet the complex number form. If at any point the expression under the square root becomes negative (i.e., a complex number), then we conclude that the original $\sqrt{e}$ is irrational.

Interpretation

If the quadratic solution contains complex or irrational numbers, then $\sqrt{e}$ is irrational. If the recursion resolves to purely rational roots, then $\sqrt{e} \in \mathbb{Q}$.

Method 2

This method introduces a logarithmic formula to find the **nearest bounding perfect squares** around a natural number x. It is based on the observation that perfect squares follow the form a^2, and that there exists a pattern linking perfect squares and consecutive odd numbers. This pattern forms the basis of a predictive function that identifies the perfect squares bounding any number x, and indirectly allows us to assess whether $\sqrt{x}$ is rational or irrational.

Foundational Pattern: Perfect Squares and Odd Numbers

By constructing two parallel sequences — one of perfect squares starting from 0, and one of odd numbers starting from 1 — we notice a fascinating relationship:

$$(0, 1),\ (1, 3),\ (4, 5),\ (9, 7),\ (16, 9),\ (25, 11),\ \ldots$$

Here, each pair (m, d) consists of a perfect square m and the subsequent odd number d. A key observation is that the sum of a perfect square and the next odd number yields the next perfect square. The difference between two consecutive perfect squares is always an odd number:

$$m^2 - n^2 = 2n + 1 \iff m = n + 1$$

From the equation we can tell that 2n is indeed an even number and adding 1 to it makes it an odd number. Thus, we can group the number line into indexed sets of the form (m, d), with perfect square and odd number pairs.

Finding the Position of a Number x

The question becomes: given a natural number x, how can we determine which pair (m, d) surrounds it?

We know that the n-th odd number is given by:

$$d_n = 2n - 1$$

To find the odd number of the pair (m, d) for the given number, x, we solve for n such that:

$$2n - 1 = x \quad \Rightarrow \quad n = \frac{x + 1}{2}$$

At this point, further calculations would be required to determine the specific odd number in question. However, to avoid excessive or unnecessary computations, we instead proceed by expressing the relationship using logarithmic functions leading us straight to what we want.

Logarithmic Insight and Function Derivation

Because odd numbers increase by 2, and perfect squares follow the exponential pattern a^2, we approximate the position of x in the sequence using base-2 logarithms. Taking logarithms reverses the exponential structure, allowing us to estimate the bounding indices.

Let:

$$d = 2\log_2 x + 1$$

To calculate the inferior perfect square m from the odd number d, we use:

$$m = \left(\frac{d-1}{2}\right)^2$$

Now, replace d using the logarithmic estimate $d = 2\log_2 x + 1$. When simplified and rewritten using logarithmic identities, the inferior perfect square becomes

$$(\lfloor \log_2 x \rfloor)^2$$

, and to calculate the superior perfect square, $(m + d)$ and the general function becomes:

$$f(x) = \left((\lfloor \log_2 x \rfloor)^2,\ (\lfloor \log_2 x \rfloor + 1)^2 \right)$$

This formula gives the two **nearest bounding perfect squares** that enclose the number x. It is both analytical and structural — combining properties of logarithms and number sequences.

Detecting Rationality of $\sqrt{x}$

Now, given:

$$f(x) = (a^2,\ (a+1)^2)$$

we infer:

If x lies **strictly between** these two values, then $\sqrt{x}$ is not an integer, $x \notin \mathbb{Z}$, and more precisely, it is **irrational**. If $x = a^2$, then it is a perfect square, and $\sqrt{x} = a \in \mathbb{Q}$. This method differs from simply taking $\sqrt{x}$ and checking if it's whole, because it structurally analyzes the number's position between squares using a logarithmic lens.

Example

Let $x = 17$. Then:

$$\lfloor \log_2(17) \rfloor = 4, \quad f(17) = (16,\ 25)$$

Since $17 \exists between(16, 25)$ but is not equal to either, it lies strictly between two perfect squares. Therefore, $\sqrt{17} \notin \mathbb{Q}$, and is irrational.

Remark

Unlike methods that use direct square root comparison or factorization, this logarithmic technique provides a systematic way to find bounding squares using basic arithmetic and logarithms. It is also sensitive to the edge case where the input x is a perfect square itself — correctly identifying that the square root is rational in that case.

The Bounding Function

Define:

$$f(x) = \left((\lfloor \log_2 x \rfloor)^2, (\lfloor \log_2 x \rfloor + 1)^2 \right)$$

This function gives the perfect square just below x, and the next perfect square just above it — based on powers of two.

Rationality Inference

Let $x \in \mathbb{N}$. Compute $f(x)$.

- If x **is itself a perfect square**, then:

$$x = (\lfloor \log_2 x \rfloor)^2$$

which implies $\sqrt{x} \in \mathbb{Q}$

- Otherwise, x lies strictly between two perfect squares:

$$f(x) = (a^2, b^2), \quad \text{where } a^2 < x < b^2$$

which implies $\sqrt{x} \notin \mathbb{Q}$

Example

Let $x = 17$. Then:

$$\lfloor \log_2(17) \rfloor = 4, \quad f(17) = (16, 25)$$

So 17 lies between 16 and 25. Hence, $\sqrt{17}$ is not an integer, and since no rational number squared equals 17, we conclude $\sqrt{17}$ is irrational.

Alternate Method and Equation Equivalence

There exists an alternative computational method to determine the immediate inferior and superior perfect squares surrounding a natural number x. This traditional method involves taking the square root of x, applying the floor function to find the closest lower integer value, and then calculating:

$$(\lfloor \sqrt{x} \rfloor)^2 \quad \text{and} \quad (\lfloor \sqrt{x} \rfloor + 1)^2$$

as the nearest inferior and superior perfect squares, respectively.

To analyze this further, I constructed an equation by observing the relationship:

$$(\sqrt{x} - 1)^2 = (\log_2 x)^2 \iff x \in \{n^2 | n \in \mathbb{Q}\}$$

which, upon simplification, becomes:

$$\log_2 x = \sqrt{x} - 1 \iff x \in \{n^2 | n \in \mathbb{Q}\}$$

Raising both sides as exponents of 2 to eliminate the logarithm gives:

$$2^{\sqrt{x}-1} = x \iff x \in \{n^2 | n \in \mathbb{Q}\}$$

This leads to the compact form:

$$2^{x^{1/2} - x^0} = x \iff x \in \{n^2 | n \in \mathbb{Q}\}$$

This expression presents an interesting equivalence between exponential and logarithmic perspectives of the bounding square problem. However, it is important to note that this identity only holds true when $x \notin \mathbb{Q}^2$; that is, x is not a perfect square. If x is a perfect square, then $\sqrt{x} \in \mathbb{Q}$, and the original formulation would not reflect the intended asymmetry of the bounding structure.

Odd Number Identification via Position Function

A natural number x is odd if and only if it can be written in the form:

$$x = 2n - 1 \quad \text{for some } n \in \mathbb{N}$$

This structure means that odd numbers occur at positions indexed by n in a sequential pattern. Reversing the expression gives:

$$n = \frac{x+1}{2}$$

This implies the following:

- If $\frac{x+1}{2} \in \mathbb{N}$, then x is odd.
- If x is even, then $\frac{x+1}{2} \notin \mathbb{N}$, and x cannot be odd.

This insight is especially useful when analyzing the parity of numbers within larger algebraic or computational structures. It provides an immediate, structural way to confirm oddness or reject it purely algebraically.

Conclusion

These two original tests offer accessible formula-based approaches to rationality testing:

- The first method uses algebraic transformations and recursive discriminants to track the rationality of the roots.
- The second method uses the logarithmic approximation to detect irrational square roots via bounding perfect squares.

Together, they demonstrate how number theory and algebra can merge with computational insight to produce intuitive tests for the foundational properties of numbers.